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Giant waves

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It is suggested that giant waves, as observed on the Agulhas Current, occur where the wave groups are reflected by the current. The local behaviour of the wave amplitude is modelled by the nonlinear Schrödinger equation

$$ia_\tau = a_{\rho\rho} - \rho a + \beta|a|^2 a.$$

For waves of a given incident wave amplitude the steady solutions are stable.

1. Introduction

During the closure of the Suez Canal a number of ships, particularly oil tankers, have reported extensive damage caused by giant waves off the south-east coast of South Africa (Mallory 1974; Sturm 1974; Sanderson 1974). Two particularly unfortunate vessels are the *World Glory*, which broke in two and sank in June 1968, and the *Neptune Sapphire*, which lost 60 m of its bow section in August 1973. We can only speculate that giant waves may account for many of the ships which have been lost without trace off this coast. When returning from the Persian Gulf the tankers take advantage of the rapid Agulhas Current, and all except one of the eleven incidents listed by Captain Mallory (1974) involved vessels riding on the current. By examining weather charts, Mallory showed that when the incidents occurred the dominant wind-produced waves were opposed by the current.

Even the longest swell can be regarded as being short relative to the horizontal scale of current variations. Thus we can expect a ray solution to give a good overall description of how the wave field is modified by the current (Longuet-Higgins & Stewart 1961). For waves of small steepness this leads to consideration of the local dispersion relation

$$\omega = \mathbf{U}.\mathbf{k} + \Omega(\mathbf{x}, \mathbf{k}).$$

Here ω is the wave frequency, $\mathbf{k}$ the local wavenumber, $\mathbf{U}$ the (effectively) depth-independent current, and Ω the dispersion relation in the absence of the current. For deep-water waves $\Omega = (g|\mathbf{k}|)^{\frac{1}{2}}$. According to the ray solution there is not separate conservation of wave and stream energies (Longuet-Higgins & Stewart 1961). However, the wave action (local wave energy density divided by Ω) propagates at the velocity $\mathbf{U} + \partial\Omega/\partial\mathbf{k}$ of a wavegroup (Bretherton & Garrett 1968). In particular, this implies that when the waves are opposed by the current the wave height will be relatively large as not only is the group velocity reduced, but also the waves receive energy from the current (i.e. Ω is increased). A more spectacular enhancement of wave heights is hinted at if there should exist a (possibly moving) curve at which the wave groups are reflected. At such a caustic of the ray paths the ray solution predicts a singularity in the wave height. Of course, the ray solution is not valid near such singularities (McKee 1974). However, in storm conditions even a doubling of the wave height would be quite traumatic. Thus we are led to speculate that when waves on an opposing current come across a reflexion line giant waves are caused (Peregrine 1976).

Although giant waves are far from being infinitesimal, we can hope that many aspects of their behaviour can be modelled, at least qualitatively, by the equations which apply when the wave amplitude is such that the nonlinearity has an order-one effect. An estimate of this amplitude range can be obtained by comparing the displacement of the reflexion line due to finite amplitude effects with the width of the highly amplified region. Let ϵ^3 denote the ratio of a typical wavelength to a typical length scale for current or depth variations. From linear theory the width of the highly amplified region is of order ϵ^2 (McKee 1974), while from nonlinear theory the displacement of the reflexion line due to the increased group velocity is of the order of the wave steepness squared (Peregrine & Thomas 1976). Thus, for linear and nonlinear effects to be comparable the wave steepness near the caustic must be of order ϵ. We note that this corresponds to the waves far from the caustic having a steepness of order $\epsilon^{\frac{3}{2}}$.

The above scaling assumptions are not unrealistic for giant-wave conditions on the Agulhas Current. The incoming swell typically has a wavelength of 200 m and a steepness of 0·08 in the open ocean (Mallory 1974), and a typical width of the current is 140 km. The ratio of lengths gives $\epsilon \doteq \frac{1}{9}$, and the steepness of the incoming wave is not unreasonably large relative to the order-of-magnitude estimate $\epsilon^{\frac{3}{2}} = \frac{1}{27}$. These values suggest that there would be about a threefold amplification of the wave height near the caustic. Thus the crest-to-trough wave height could be as much as 15 m. Locally generated waves superimposed upon such a giant wave could aggravate the situation by a further 5 m. The scaling assumptions are less pertinent for a moving caustic associated with the shorter and steeper locally generated waves.

Unfortunately, such would be the complexity of a calculation valid for weakly nonlinear waves near a moving caustic on a rotational stratified non-uniform current that it justifies restricting attention to a simple case. Thus in §§ 2 and 3 we first study deep-water waves close to a stationary caustic which is perpendicular to a steady unidirectional irrotational current. The choice of problem and much of the analysis is guided by Davey & Stewartson's (1974) work on the evolution of almost plane waves. The properties of a model equation are considered in § 4. Finally, §§ 5 and 6 are directed towards the possibility of using the results of the paper for more realistic situations. Complementary studies for infinitesimal waves near curved moving caustics and nonlinear waves in a region bounded away from the reflexion line are given by Smith (1975) and Peregrine & Thomas (1976).

2. Derivation of the model equations

Let U and L respectively be typical values of the current velocity and the horizontal length scale of current variations. If deep-water waves with wavelengths of order $\epsilon^3 L$ are to be reflected by the current, then it is appropriate to define

$$\epsilon = (U^2/gL)^{\frac{1}{3}}.$$

For definiteness we shall assume that the vertical scale is L, but we note that the major results of this section remain valid provided that the depth scale greatly exceeds the wavelength. In dimensionless form the velocity potential Φ and the small undulations $\epsilon^3\zeta$ of the undisturbed current satisfy the equations

$$\left.\begin{array}{l}\nabla^2\Phi = 0, \quad \partial\zeta/\partial t + \nabla\Phi . \nabla\zeta = \epsilon^{-3}\partial\Phi/\partial z \quad \text{on} \quad z = \epsilon^3\zeta, \\ \zeta + \partial\Phi/\partial t + \frac{1}{2}(\nabla\Phi)^2 = 0 \quad \text{on} \quad z = \epsilon^3\zeta, \quad \partial\Phi/\partial z + \nabla\Phi . \nabla h = 0 \quad \text{on} \quad z = -h,\end{array}\right\} \quad (1)$$

where h is the water depth and ∇ is the three-dimensional gradient operator $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

From (1) we infer that Φ and ζ have asymptotic expansions of the form

$$\Phi = \Phi_0 + \epsilon^3\Phi_3 + O(\epsilon^6), \quad \zeta = \zeta_0 + O(\epsilon^3),$$

where the Φ_j and ζ_j are all independent of ϵ. Employing a Taylor series in z to transfer the free-surface boundary conditions to the horizontal surface $z = 0$ and then extracting like powers of ϵ^3 from (1), we can obtain a sequence of field equations and boundary conditions which are all independent of ϵ. In particular, if the current is independent of both y and t, then on $z = 0$ we have

$$\partial\Phi_0/\partial z = 0, \quad \zeta_0 = -\tfrac{1}{2}U^2, \quad \partial^2\Phi_0/\partial z^2 = -dU/\partial x,$$

where U is the value of $\partial\Phi_0/\partial x$ on $z = 0$. (More general currents are considered in §5.)

The nonlinear equations for the waves can be obtained from (1) by replacing ζ by $\zeta + \epsilon\eta$, Φ by $\Phi + \epsilon^4\phi$ and using the Taylor-series operator

$$1 + \epsilon^4\eta\partial/\partial z + \tfrac{1}{2}\epsilon^8\eta^2(\partial/\partial z)^2 + \dots$$

to transfer the free-surface boundary conditions from $z = \epsilon^3\zeta + \epsilon^4\eta$ to $z = \epsilon^3\zeta$.

The aim of this section is to derive an approximate solution for the waves which is valid within a horizontal distance of order ϵ^2 of the reflexion line and within a vertical distance of order ϵ^3 of the free surface (i.e. where the wave amplitude is largest). If these were the only short scales involved, then for a boundary-layer type of solution it would suffice to introduce stretched coordinates (Nayfeh 1973, chap. 4)

$$X = \epsilon^{-2}x, \quad Z = \epsilon^{-3}z - \zeta \quad (2a)$$

and to use local power-series representations for the undisturbed current,

e.g. $$U = U^{(0)} + \epsilon^2 X U^{(1)} + \tfrac{1}{2}\epsilon^4 X^2 U^{(2)} + \dots.$$

However, for the present problem the separation of the individual wave crests has a length scale of only ϵ^3. Furthermore, any unsteadiness is swept along the caustic at the group velocity, taking a time of order ϵ to traverse the highly amplified region. To cope with these additional short scales we introduce the phase, convected and evolution co-ordinates (Davey & Stewartson 1974)

$$\theta = \epsilon^{-3}(-\omega t + ly + k(\epsilon)\,x), \quad Y = \epsilon^{-2}(y - c(\epsilon)\,t), \quad T = \epsilon^{-1}t. \tag{2b}$$

Thus, in the nonlinear equations for the waves we formally express $\partial/\partial t$, ∇ and $\partial/\partial z$ in terms of the undetermined functions $k(\epsilon)$ and $c(\epsilon)$ and the derivative operators $\partial/\partial\theta$, $\partial/\partial X$, $\partial/\partial Y$, $\partial/\partial Z$ and $\partial/\partial T$. In the spirit of the method of multiple scales, we shall require that the waves are 2π-periodic with respect to θ (Nayfeh 1973, chap. 6).

The resulting version of the equations for the waves is

$$[k^2+l^2]\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial Z^2} + 2\epsilon\left\{k\frac{\partial^2\phi}{\partial X\partial\theta} + l\frac{\partial^2\phi}{\partial Y\partial\theta}\right\} + \epsilon^2\left\{\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2}\right\} = O(\epsilon^3), \tag{3a}$$

$$\begin{aligned}(kU^{(0)}-\omega)\frac{\partial\eta}{\partial\theta} &+ \epsilon\left\{U^{(0)}\frac{\partial\eta}{\partial X} - c\frac{\partial\eta}{\partial Y} + [k^2+l^2]\frac{\partial\phi}{\partial\theta}\frac{\partial\eta}{\partial\theta}\right\}\\ &+ \epsilon^2\left\{\frac{\partial\eta}{\partial T} + XkU^{(1)}\frac{\partial\eta}{\partial\theta} + k\left(\frac{\partial\phi}{\partial\theta}\frac{\partial\eta}{\partial X} + \frac{\partial\phi}{\partial X}\frac{\partial\eta}{\partial\theta}\right) + l\left(\frac{\partial\phi}{\partial\theta}\frac{\partial\eta}{\partial Y} + \frac{\partial\phi}{\partial Y}\frac{\partial\eta}{\partial\theta}\right)\right.\\ &\left. + [k^2+l^2]\eta\frac{\partial\eta}{\partial\theta}\frac{\partial^2\phi}{\partial\theta\partial Z}\right\} = \frac{\partial\phi}{\partial Z} + \epsilon\eta\frac{\partial^2\phi}{\partial Z^2} + \frac{\epsilon^2\eta^2}{2}\frac{\partial^3\phi}{\partial Z^3} + O(\epsilon^3) \quad \text{on} \quad Z = 0,\end{aligned} \tag{3b}$$

$$\begin{aligned}\eta + (kU^{(0)}-\omega)\frac{\partial\phi}{\partial\theta} &+ \epsilon\left\{U^{(0)}\frac{\partial\phi}{\partial X} - c\frac{\partial\phi}{\partial Y} + \tfrac{1}{2}[k^2+l^2]\left(\frac{\partial\phi}{\partial\theta}\right)^2\right.\\ &\left. + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial Z}\right)^2 + (kU^{(0)}-\omega)\,\eta\frac{\partial^2\phi}{\partial\theta\partial Z}\right\}\\ &+ \epsilon^2\left\{\frac{\partial\phi}{\partial T} + XkU^{(1)}\frac{\partial\phi}{\partial\theta} + \tfrac{1}{2}(kU^{(0)}-\omega)\,\eta^2\frac{\partial^3\phi}{\partial\theta\partial Z^2} + U^{(0)}\eta\frac{\partial^2\phi}{\partial X\partial Z} + k\frac{\partial\phi}{\partial X}\frac{\partial\phi}{\partial\theta}\right.\\ &\left. + l\frac{\partial\phi}{\partial Y}\frac{\partial\phi}{\partial\theta} + [k^2+l^2]\,\eta\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial\theta\partial Z} + \eta\frac{\partial\phi}{\partial Z}\frac{\partial^2\phi}{\partial Z^2}\right\} = O(\epsilon^3) \quad \text{on} \quad Z = 0.\end{aligned} \tag{3c}$$

As is usual in such boundary-layer calculations, the far-field boundary condition at the seabed is relaxed to a condition of no exponential growth for large negative Z.

The 2π-periodicity requirement makes it natural for us to use Fourier-series representations for ϕ and η:

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n \exp(in\theta) \quad \text{with} \quad \phi_{-n} = \phi_n^*.$$

The ϕ_1 and η_1 terms correspond to infinitesimal waves and the other terms can be regarded as being forced by the nonlinearities in the free-surface boundary conditions (3*b*, *c*). For the higher Fourier components this forcing has a length scale of the order of a wavelength and, in the absence of resonance, the response has the same magnitude as the forcing. However, for the θ-independent mode the forcing has the longer length scale of the wave envelope and, even in the absence of resonance, the ϕ_0 response is one order larger than the corresponding nonlinear terms in the free-surface boundary conditions. Davey & Stewartson (1974) show

how this enhancement of the ϕ_0 term precludes the evolution of almost plane nonlinear waves in water of moderate depth from being described by a single nonlinear partial differential equation. Here we have the extra complications of there being a caustic and a current; hence the compensating simplification of the water being very deep.

To obtain a regular perturbation solution of (3) when ϵ is small we put

$$k = k_0 + \epsilon k_1 + \epsilon^2 k_2 + \dots, \quad c = c_0 + \epsilon c_1 + \dots,$$
$$\phi_0 = \epsilon\phi_{01} + \dots, \qquad \phi_n = \epsilon^{n-1}\phi_{nn-1} + \dots, \text{etc.}$$

Not unexpectedly, the leading-order terms in (3) give the eigenvalue problem for infinitesimal deep-water waves. The solution is

$$\omega = k_0 U^{(0)} + \kappa^{\frac{1}{2}}, \quad \eta_{10} = i\kappa^{\frac{1}{2}} A, \quad \phi_{10} = A\exp(\kappa Z) \quad \text{with} \quad \kappa^2 = k_0^2 + l^2, \tag{4a}$$

where the amplitude factor $A(X,\ Y,\ T)$ is to be determined. The equations for ϕ_{11} and η_{11} are inhomogeneous versions of the leading-order eigenvalue problem, and therefore can only have a solution provided that the inhomogeneous terms satisfy an integrability condition. The A, $\partial A/\partial X$ and $\partial A/\partial Y$ coefficients in this condition yield the equations

$$k_1 = 0, \quad U^{(0)} + \tfrac{1}{2}k_0\kappa^{-\frac{3}{2}} = 0, \quad c_0 = \tfrac{1}{2}l\kappa^{-\frac{3}{2}}, \tag{4b}$$

i.e. the normal component of the group velocity is zero at the caustic and c_0 equals the transverse group velocity. The solutions for ϕ_{11} and η_{11} are

$$\phi_{11} = -i\kappa^{-1}\left(k_0\frac{\partial A}{\partial X} + l\frac{\partial A}{\partial Y}\right) Z\exp(\kappa Z), \quad \eta_{11} = \tfrac{1}{2}\kappa^{-\frac{3}{2}}\left(k_0\frac{\partial A}{\partial X} + l\frac{\partial A}{\partial Y}\right), \tag{4c}$$

where we have suppressed a possible multiple A_1 of the eigensolution. It is noteworthy that these terms are zero in the absence of a gradient of A. This means that the shape of the waves differs from that of a uniform wave train.

The double-frequency order-ϵ equations can be solved by inspection:

$$\phi_{21} = 0, \quad \eta_{21} = -\kappa^2 A^2. \tag{4d}$$

It happens that for deep-water waves the zero-frequency forcing terms cancel out at order ϵ. In view of the remarks made in the previous paragraph, we take the solution to be

$$\phi_{01} \text{ independent of } Z, \quad \eta_{01} = 0.$$

Finally, at order ϵ^2 the integrability condition for the ϕ_{12} and η_{12} terms provides us with the evolution equation

$$2i\kappa^{\frac{1}{2}}\frac{\partial A}{\partial T} + \frac{1}{2\kappa}\left(\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2}\right) - \frac{3}{4\kappa^3}\left(k_0\frac{\partial}{\partial X} + l\frac{\partial}{\partial Y}\right)^2 A - 2XU^{(1)}k_0\kappa^{\frac{1}{2}}A - 4\kappa^4|A|^2 A = 0, \tag{5}$$

where, without loss of generality, we have eliminated additional A and $\partial A/\partial Y$ terms by setting $c_1 = k_2 = 0$.

3. An alternative derivation

The above analysis has the virtue of being systematic. It makes quite clear the assumptions underlying its applicability and even permits higher approximations to be calculated. It does however have the drawback of being very specific. Here we present a derivation of entirely the opposite character.

We introduce the local nonlinear dispersion relation for sinusoidal waves in a frame of reference moving across the current at velocity c_0:

$$\omega = Uk - c_0 l + \Omega(x, k, l, |A|^2).$$

Let ω and (k_0, l_0) satisfy the conditions for there to be a linear-theory reflexion line along $x = 0$ and for c_0 to be the group velocity along that line:

$$\omega = U|_0 k_0 - cl_0 + \Omega(0, k_0, l_0, 0), 0 = U|_0 + [\partial\Omega/\partial k]_0, \quad 0 = -c_0 + [\partial\Omega/\partial l]_0.$$

Then for waves of frequency $\omega + \hat{\omega}$, wavenumber $(k_0 + \hat{k}, l_0 + \hat{l})$ and amplitude A we have the perturbation dispersion relation

$$\hat{\omega} = \tfrac{1}{2}\hat{k}^2 \frac{\partial^2\Omega}{\partial k^2}\bigg|_0 + \hat{k}\hat{l}\frac{\partial^2\Omega}{\partial k\partial l}\bigg|_0 + \tfrac{1}{2}\hat{l}^2\frac{\partial^2\Omega}{\partial l^2}\bigg|_0 + x\left(k_0\frac{\partial U}{\partial x}\bigg|_0 + \frac{\partial\Omega}{\partial x}\bigg|_0\right) + |A|^2\frac{\partial\Omega}{\partial|A|^2}\bigg|_0,$$

where we have retained the leading-order contributions in wavenumber, position and amplitude. The corresponding differential equation is

$$2i\frac{\partial A}{\partial t} + \frac{\partial^2\Omega}{\partial k^2}\bigg|_0\frac{\partial^2 A}{\partial x^2} + 2\frac{\partial^2\Omega}{\partial k\partial l}\bigg|_0\frac{\partial^2 A}{\partial x\partial y} + \frac{\partial^2\Omega}{\partial l^2}\bigg|_0\frac{\partial^2 A}{\partial y^2} - 2x\left(k_0\frac{\partial U}{\partial x}\bigg|_0 + \frac{\partial\Omega}{\partial x}\bigg|_0\right)A - 2\frac{\partial\Omega}{\partial|A|^2}\bigg|_0|A|^2A = 0 \qquad (6)$$

(i.e. we can associate small frequency and wavenumber changes with time and space derivatives of the complex wave amplitude). For deep-water waves the nonlinear dispersion relation is implicitly given by

$$\Omega = g^{\frac{1}{2}}[k^2 + l^2]^{\frac{1}{4}} + 2|A|^2[k^2 + l^2]^2/\Omega + O(|A|^4).$$

Using this expression in (6) and converting to non-dimensional variables we recover (5).

We note that it is only at the last stage in the argument that use was made of the fact that we are studying deep-water waves rather than shallow-water waves or some entirely different class of wave motion. Thus we can expect that (6) will apply in diverse physical contexts.

4. Steady solutions and their stability

As we should expect in view of the generality of the above analysis, the nonlinear Schrödinger equation (5) or (6) combines the features of other model equations. For example, when the nonlinear term is negligible (5) admits an Airy-function solution

$$A = \mathrm{Ai}(2X[U^{(1)}\kappa^{\frac{3}{2}}k_0/(2l^2 - k_0^2)]^{\frac{1}{3}})$$

(Smith 1975). Also, when the X term is negligible (5) is the deep-water limit of the Davey–Stewartson equation, with the minor modification that the axes are

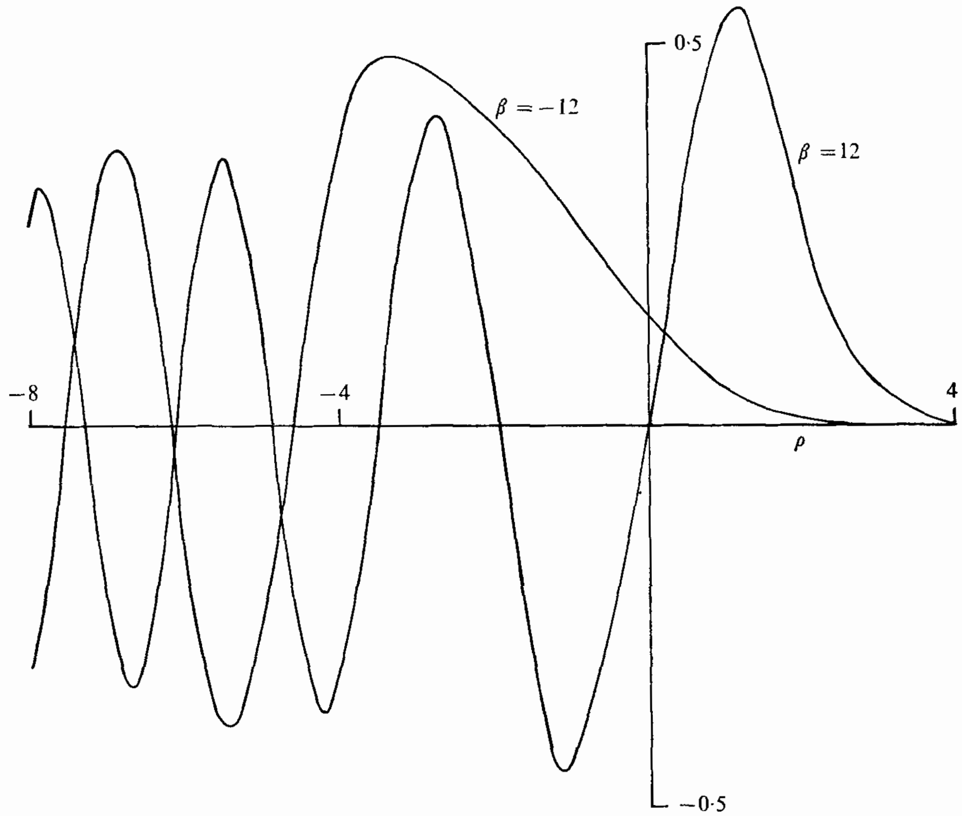

FIGURE 1. Painlevé transcendents of the second kind for $\beta = \pm 12$.

not aligned along and perpendicular to the wave crests:

$$2i\kappa^{\frac{1}{2}}\frac{\partial A}{\partial T} - \frac{1}{4\kappa}\left(\cos\alpha\frac{\partial}{\partial X} + \sin\alpha\frac{\partial}{\partial Y}\right)^2 A + \frac{1}{2\kappa}\left(-\sin\alpha\frac{\partial}{\partial X} + \cos\alpha\frac{\partial}{\partial Y}\right)^2 A = 4\kappa^2|A|^2 A$$

(equation (2.18) of Davey & Stewartson 1974). Here α is the angle between the wave crests and the X axis. This limiting case is of particular note as it suggests that the solutions to (5) might be subject to the Benjamin–Feir (1967) instability.

For simplicity we shall confine our attention to the special case in which the wave amplitude A is independent of the convected co-ordinate Y. Further, we rescale (6), or its complex conjugate, with respect to the incident wave amplitude and the linear-theory length and time scales. Specifically, if the leading-order (and uniformly valid) linear-theory solution is

$$A = \phi_\infty F(x)\,\mathrm{Ai}(\rho(x)) \tag{7a}$$

(Smith 1975), then to find the local nonlinear solution we define

$$\left.\begin{aligned} A &= \phi_\infty F|_0\, a(\rho,\tau), \quad \tau = \tfrac{1}{2}t\left(\frac{d\rho}{dx}\bigg|_0\right)^2\left|\frac{\partial^2\Omega}{\partial k^2}\bigg|_0\right|, \\ \beta &= -2\phi_\infty^2 F_0^2\frac{\partial\Omega}{\partial|A|^2}\bigg|_0\left\{\frac{\partial^2\Omega}{\partial k^2}\bigg|_0\left(\frac{d\rho}{dx}\bigg|_0\right)^2\right\}^{-1}. \end{aligned}\right\} \tag{7b}$$

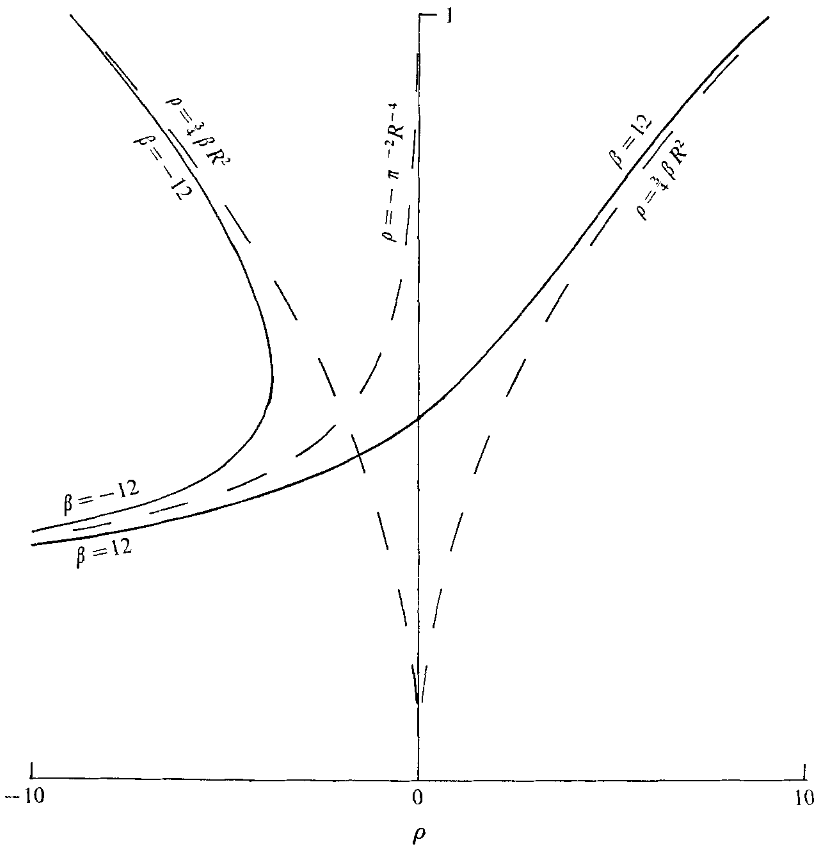

FIGURE 2. Nonlinear ray solution and asymptotes for $\beta = \pm 12$.

This leads to the one-parameter family of equations

$$ia_\tau = a_{\rho\rho} - \rho a + \beta|a|^2 a, \quad a \sim 0 \quad \text{as} \quad \rho \to \infty, \qquad (8a,b)$$

$$|\rho|^{\frac{1}{4}} a + i|\rho|^{-\frac{1}{4}} a_\rho \sim \pi^{-\frac{1}{2}} \exp\left(\tfrac{2}{3} i|\rho|^{\frac{3}{2}} + (i\beta/2\pi)\ln|\rho| + i\,\text{constant}\right) \quad \text{as} \quad \rho \to -\infty. \quad (8c)$$

For deep-water waves β is positive or negative according to whether α is less than or greater than 35°. The radiation condition can be derived from a far-field analysis of the differential equation, on the assumptions that the incident wave amplitude is steady and that the long-term averages of the incident and reflected wave-action fluxes exactly balance (see appendix). The arbitrary constant in the phase can be used to ensure that the steady solutions are real (e.g. the appropriate value is $\frac{1}{4}\pi$ for the linear case $\beta = 0$).

The steady solutions of (8) are Painlevé transcendents of the second kind (Davis 1962, chap. 7). Qualitatively the solutions resemble Airy functions with the transition from sinusoidal to exponential behaviour slightly displaced from $\rho = 0$ (see figure 1). It is only for extremely large values of β that there is any significant change in the maximum wave amplitude from that predicted by the linear-theory Airy-function solution. Even for $\beta = -12$ the maximum is reduced by less than 10%.

The above results contradict Holliday's (1973) suggestion that finite amplitude effects remove the wave barrier. The error in Holliday's conservation-equation analysis is that the mean total fluxes of mass and energy are evaluated on the hypothesis that there is no reflected wave.

A more subtle difficulty arises if the waves are analysed using Whitham's (1965) nonlinear method of averaging. The leading order approximations to the amplitude $R(\rho, \beta)$ and the phase $\psi(\rho, \beta)$ are given by

$$\rho = \tfrac{3}{4}\beta R^2 - \pi^{-2}R^{-4}, \quad \psi_\rho = -\pi^{-1}R^{-2}$$

(see figure 2). For negative β there is a real solution only for

$$\rho \leqslant -\tfrac{3}{4}(-3\beta/\pi)^{\frac{2}{3}},$$

and the position where the graph becomes vertical can be regarded as being an estimate both of the displaced caustic position and of the maximum wave height. However, for a positive β the solution does not give a clear indication of its inapplicability for large positive ρ. The reason for this shortcoming of Whitham's method is that it takes no account of the far-field boundary condition and provides us with an approximation to the Bi-like solution rather than the required Ai-like solution. This trouble can be alleviated if the Whitham-method solution is interpreted as an outer solution and is matched with the inner solutions discussed above (Peregrine & Thomas 1976).

An infinitesimal stability analysis of the real steady solutions of (8) leads to consideration of the eigenvalue problem

$$b_{\rho\rho} - \rho b + \lambda b + \beta a^2(2b + b^*) = 0,$$

$$b \sim 0 \quad \text{as} \quad \rho \to \infty, \quad |\rho|^{\frac{1}{4}}b + i|\rho|^{-\frac{1}{4}}b_\rho \sim 0 \quad \text{as} \quad \rho \to -\infty.$$

The steady solution is unstable to infinitesimal disturbances if there exists any eigensolution with the imaginary part of λ negative. The far-field forms of the solutions to the differential equation for b together with the radiation condition (of zero incident wave amplitude) suggest that there is stability.

To demonstrate this it suffices to show that there cannot exist a neutrally stable eigenmode (i.e. a mode with λ real). For ρ large and negative we first put

$$\xi = \tfrac{2}{3}|\rho - \lambda|^{\frac{3}{2}}, \quad a = |\rho - \lambda|^{-\frac{1}{4}}G(\xi), \quad b = |\rho - \lambda|^{-\frac{1}{4}}H(\xi).$$

Thus G is of order unity for large ξ, and $H(\xi)$ satisfies the equations

$$\frac{d^2H}{d\xi^2} + H + \frac{2\beta}{3\xi}(2H + H^*)G^2 - \frac{5}{36}\frac{H}{\xi^2} = 0,$$

$$H - i\,dH/d\xi \sim 0 \quad \text{as} \quad \xi \to \infty.$$

Next, we formally introduce the slow co-ordinate $\chi = \ln \xi$ and regard G and H as functions of both ξ and χ (Nayfeh 1973, chap. 6). This leads to the partial differential equations

$$\frac{\partial^2 H}{\partial \xi^2} + H + e^{-\chi}\left\{\frac{2\partial^2 H}{\partial\xi\partial\chi} + \frac{2}{3}\beta G^2(2H + H^*)\right\} + e^{-2\chi}\left\{\frac{\partial^2 H}{\partial\chi^2} - \frac{\partial H}{\partial\chi} - \frac{5}{36}H\right\} = 0,$$

$$H - i\partial H/\partial\xi \sim 0 \quad \text{as} \quad \chi \to \infty,$$

where we require that H is 2π-periodic with respect to ξ [the periodicity of G being ensured by means of (8)]. For large χ the leading-order terms have the solution

$$H_0 = p(\chi)\, e^{i\xi} + q(\chi)\, e^{-i\xi}, \quad \text{where} \quad p \sim 0 \quad \text{as} \quad \chi \to \infty.$$

At the next order the non-secularity condition yields the coupled linear differential equations

$$i\frac{dp}{d\chi} + \frac{\beta}{3\pi}(2p+q^*) = 0, \quad -i\frac{dq}{d\chi} + \frac{\beta}{3\pi}(2q+p^*) = 0.$$

The solutions for p and q are linear combinations of the oscillatory exponentials $\exp(\pm i\chi\beta/3^{\frac{1}{2}}\pi)$. The solution is compatible with the radiation condition only if $p = q = 0$. Thus we infer that H_0 and hence b is identically zero, i.e. that there does not exist a neutrally stable mode. Numerical solutions to (8) have confirmed that the steady solutions are stable even to large disturbances for both signs of β.

5. Applicability

The analysis of § 2 concerns a highly idealized situation. We now consider whether it is feasible to obtain a similarly detailed solution for the waves when we relax the conditions that the caustic is stationary and straight and that the current is irrotational.

With respect to axes moving at the local caustic velocity the caustic is automatically stationary. However, the relative current speed (or even direction) becomes time-dependent. For the solution for the waves to be unchanged to the order of the calculations presented above, it suffices that the changes in caustic velocity and the movement of the caustic across the current take place on a time scale in excess of ϵ^{-2} wave periods. This means that the nonlinear caustic solution has sufficient time to adjust to changes in the wave or current conditions. On the Agulhas Current the swell typically has a period of only 12 s, whereas it takes several hours either for a wave group to traverse the current or for there to be a noticeable change in the direction of frequency of the swell. Thus, as regards the local structure of the waves, we can safely relax the stationarity requirement.

For a curved caustic we must allow both for the curvature and for the fact that the current need not be perpendicular to the caustic. If $R(y, t)$ is the radius of curvature and V is the component of velocity across the caustic, then to the order of the calculations given in § 2 it suffices that we replace l, $\partial/\partial Y$, kU and $U\partial/\partial X$ respectively by

$$\frac{l}{1-\epsilon^2 X/R}, \quad \frac{1}{1-\epsilon^2 X/R}\frac{\partial}{\partial Y}, \quad kU + \frac{lV}{1-\epsilon^2 X/R}, \quad U\frac{\partial}{dX} + \frac{V}{1-\epsilon^2 X/R}\frac{\partial}{\partial Y}.$$

The only changes to (4a–d) are

$$\omega = k_0\, U^{(0)} + l\, V^{(0)} + \kappa^{\frac{1}{2}}, \quad c_0 = V^{(0)} + \tfrac{1}{2} l \kappa^{-\frac{3}{2}},$$

while the X term in (5) is changed to

$$-2XA\kappa^{\frac{1}{2}}\{k_0 U^{(1)} + lV^{(1)} + lV^{(0)}R^{-1} - \tfrac{1}{2}l^2\kappa^{-\frac{3}{2}}R^{-1}\}.$$

Weak vorticity of the current can be expected to have a correspondingly weak effect upon the nonlinear dispersion relation. Thus the heuristic analysis of § 3 permits us to justify the continued use of (5) for arbitrary wide currents. A more careful argument is needed if we wish to retain (4*a*–*d*). On the length and time scales of the wave crests the current vorticity is of order ϵ^3. It follows that the rotational part of the wave motion, due to distortions of the vortex lines, is order ϵ^3 smaller than the irrotational part. This is just small enough for the irrotational analysis of § 2 to remain applicable to the waves. Of course, the current cannot be described, even at leading order, by the irrotational equations (1).

For a curved or moving caustic there is the preliminary step of determining the position of the caustic and the amplitude of the incident waves near the caustic. This can be achieved by obtaining the linear ray description of the waves. Indeed, if the ray solutions are continued beyond the point where they touch the caustic to give a ray description of the reflected waves, then it is trivial to reconstruct the linear-theory caustic solution (Smith 1975). Unsteadiness, curvature and vorticity do not radically modify the nonlinear caustic equation. Thus, as shown in (7*a*, *b*) the conversion from a (uniformly valid) linear to a (locally valid) nonlinear caustic solution merely entails the replacing of the Airy function by the appropriate Painlevé transcendent. The parameter β will, of course, be a slowly varying function of time and position.

Caustics can occur for a variety of reasons, many of which are independent of there being a current. The marked correlation between the occurrence of destructive waves and the presence of a strong current would suggest that the current does play an important role. The simplest possibility is that incident waves at a small angle δ to the opposing current are refracted away (McKee 1974). For a parallel current $(0, -V(x), 0)$ the leading-order linear-theory solution for the value of ϕ on the free surface can be written as

$$\phi = \phi_\infty \exp(i\kappa_\infty \cos\delta y)\, F(x)\, \mathrm{Ai}(\rho(x)),$$

with

$$\tfrac{2}{3}(-\rho)^{\frac{3}{2}} = \kappa_\infty \int_{x_0}^{x} \left[\left(1 - \frac{V}{c_\infty}\cos\delta\right)^4 - \cos^2\delta\right]^{\frac{1}{2}} dx,$$

$$F(x) = (-\rho)^{\frac{1}{4}}\, 2\pi^{\frac{1}{2}} \left(1 - \frac{V}{c_\infty}\cos\delta\right)^2 \sin^{\frac{1}{2}}\delta \left[\left(1 - \frac{V}{c_\infty}\cos\delta\right)^4 - \cos^2\delta\right]^{-\frac{1}{4}}.$$

Here x_0 is the caustic position and ϕ_∞, κ_∞ and c_∞ are the values far from the current of the velocity-potential amplitude, the wavenumber and the phase velocity of the incident waves. At the caustic we have

$$V = c_\infty(1 - \cos^{\frac{1}{2}}\delta)\sec\delta, \quad -\frac{d\rho}{dx} = \left[-4\frac{\kappa_\infty^2}{c_\infty}\frac{dV}{dx}\bigg|_0\right]^{\frac{1}{3}} \cos^{\frac{5}{6}}\delta,$$

$$F|_0 = 2\pi^{\frac{1}{2}} \cos^{\frac{5}{6}}\delta \sin^{\frac{1}{2}}\delta \left[-\frac{4}{\omega}\frac{dV}{dx}\bigg|_0\right]^{\frac{1}{6}}.$$

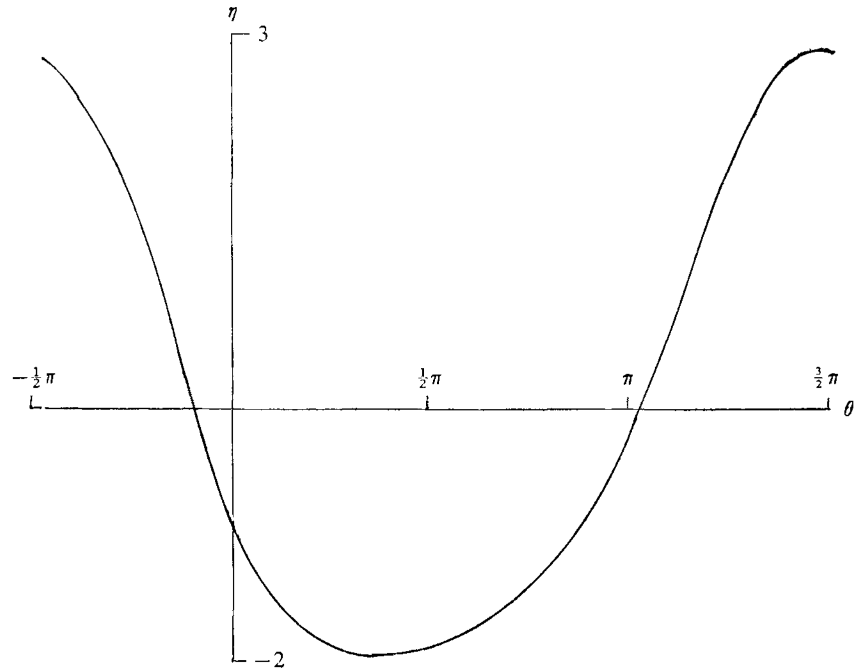

FIGURE 3. Dimensionless wave profile exhibiting slight asymmetry.

This idealized solution, due to McKee (1974), enables us to make several qualitative predictions concerning the giant waves. First, the caustic is necessarily on the seaward side of the current, and the landward side of the current only has the locally generated waves. Second, for very long swell with relatively large phase velocity there can be a caustic only for waves which are almost directly opposed by the current. In particular, for 12 s waves on a current of up to 2 ms^{-1}, δ must be less than 33°. Third, the width of the highly amplified region and the amplification factor $F|_0$ are fairly insensitive both to the flatness of the velocity profile and to δ. For 12 s waves on the Agulhas Current representative values would be

$$-dx/d\rho \sim 2\,\text{km}, \quad F|_0 = 8.$$

Thus the giant waves occupy a tiny fraction of the current width, but the wave-height amplification could exceed a factor of 4. All these predictions agree with the wave conditions when giant waves are encountered (Mallory 1974).

6. Wave profile

The hazard to shipping of the unusual waves on the Agulhas Current is due to the combination of large steepness and large wave height. If the waves were less steep then a vessel could ride over them and if the wave height were less then there would not be such a tremendous weight of water crashing onto the bow of the vessel. The mariners' reports seem to suggest that the most destructive waves have the added feature of being steepest on the foreward face (Sturm 1974;

Mallory 1974). This reduces the time available for the ships' forepart to rise to meet the oncoming wave crest. With large tankers there is a considerable premium on speed and an understandable tendency to disregard the wave conditions. Thus, having successfully risen over one large wave the ship ends up steaming (possibly at full speed) downhill and buries its bow in the next wave with disastrous effects.

According to the analysis presented in § 2 the dimensionless wave profile is given to second order by

$$\eta = -2\kappa^{\frac{1}{2}} A \sin\theta + \epsilon\{\kappa^{-\frac{3}{2}}(k_0\, \partial A/\partial X + l\, dA/\partial Y)\cos\theta - 2\kappa^2 A^2 \cos 2\theta\}.$$

The presence of the cos θ term does indeed permit the wave profile to be asymmetric. For illustrative purposes figure 3 shows the dimensionless wave profile for

$$\epsilon = \tfrac{1}{8}, \quad k_0 = l = 1, \quad A = 1, \quad \partial A/\partial X = -4.$$

This can be thought of (with allowance for an ϵ^{-1} exaggeration of the vertical scale) as typifying the waves experienced by a ship as it first encounters the caustic. Thus, almost straight away the waves are of the most destructive kind. The superposition of the shorter locally generated waves can make the asymmetry even more pronounced (Mallory 1974, figure 4).

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Appendix. Derivation of the radiation condition

Our intention is to express in mathematical terms the fact that the incident, but not the reflected wave is independent of any time-dependent behaviour near the reflexion line. An *ad hoc* derivation would be first to derive the linear-theory radiation condition and then to use the Whitham's method to estimate the nonlinear correction to the phase. Here we present a more systematic derivation based on a far-field analysis of the equation

$$i a_\tau = a_{\rho\rho} - \rho a + \beta |a|^2 a.$$

As in the stability analysis, the preliminary change of variables

$$\xi = \tfrac{2}{3}|-\rho|^{\frac{3}{2}}, \quad a = |-\rho|^{-\frac{1}{4}} G(\xi, \tau)$$

emphasizes the sinusoidal structure of the far field:

$$\frac{\partial^2 G}{\partial \xi^2} + G - i\left(\frac{2}{3\xi}\right)^{\frac{2}{3}} \frac{\partial G}{\partial \tau} + \frac{2\beta}{3\xi}|G|^2 G - \frac{5}{36}\frac{G}{\xi^2} = 0.$$

Indeed, we can immediately infer that in the far field

$$G \sim p(\xi, \tau)\, e^{i\xi} + q(\xi, \tau)\, e^{-i\xi},$$

where p and q are slowly varying functions of ξ. Thus a mathematical statement involving only the combination $G - iG_\xi$ or equivalently $|\rho|^{\frac{1}{4}} a + i|\rho|^{-\frac{1}{4}} a_\rho$ is in effect a statement about the p amplitude factor.

Further mathematical analysis is needed if we are to confirm that p is the incident wave term, and if we are to determine the systematic effect of the nonlinearity upon the phase of the waves. To do this we formally introduce the group-velocity and phase-shift co-ordinates

$$\mu = (\tfrac{3}{2}\xi)^{\frac{2}{3}}, \quad \chi = \ln \xi$$

and regard G as a function of the four variables ξ, τ, μ and χ:

$$\frac{\partial^2 G}{\partial \xi^2} + G + \left(\frac{2}{3}\right)^{\frac{2}{3}} \exp(-\tfrac{2}{3}\chi)\left\{\frac{\partial^2 G}{\partial \xi \partial \mu} - i\frac{\partial G}{\partial \tau}\right\} + \exp(-\chi)\left\{\frac{2\partial^2 G}{\partial \xi \partial \chi} + \tfrac{2}{3}\beta |G|^2 G\right\}$$
$$+ \exp(-\tfrac{4}{3}\chi)\frac{1}{9}\left(\frac{3}{2}\right)^{\frac{2}{3}}\frac{\partial^2 G}{\partial \mu^2} + \exp(-\tfrac{5}{3}\chi)\left\{\frac{\partial^2 G}{\partial \mu \partial \chi} - \frac{1}{3}\frac{\partial G}{\partial \mu}\right\}$$
$$+ \exp(-2\chi)\left\{\frac{\partial^2 G}{\partial \chi^2} - \frac{\partial G}{\partial \chi} - \frac{5}{36}G\right\} = 0.$$

For large χ we pose the series representation

$$G = G_0 + \exp(-\tfrac{1}{3}\chi)\, G_1 + \exp(-\tfrac{2}{3}\chi)\, G_2 + \dots,$$

where the $G_j(\xi, \tau, \mu, \chi)$ satisfy the sequence of equations

$$\partial^2 G_0/\partial \xi^2 + G_0 = 0, \quad \partial^2 G_1/\partial \xi^2 + G_1 = 0,$$

$$\frac{\partial^2 G_2}{\partial \xi^2} + G_2 = \left(\frac{2}{3}\right)^{\frac{2}{3}}\left\{i\frac{\partial G_0}{\partial \tau} - \frac{\partial^2 G_0}{\partial \xi \partial \mu}\right\},$$

$$\frac{\partial^2 G_3}{\partial \xi^2} + G_3 = \left(\frac{2}{3}\right)^{\frac{2}{3}}\left\{i\frac{\partial G_1}{\partial \tau} - \frac{\partial^2 G_1}{\partial \xi \partial \mu}\right\} - 2\frac{\partial^2 G_0}{\partial \xi \partial \chi} - \tfrac{2}{3}\beta |G_0|^2 G_0,$$

etc. The key hypothesis in solving these equations is that all terms are uniformly bounded with respect to the short and middle scales ξ and μ (Nayfeh 1973, chap. 6).

From the first two equations we have

$$G_0 = p_0 e^{i\xi} + q_0 e^{-i\xi}, \quad G_1 = p_1 e^{i\xi} + q_1 e^{-i\xi},$$

where p_j and q_j are independent of ξ. Next, the equation for G_2 has a bounded solution only if the terms on the right-hand side have zero $e^{i\xi}$ and $e^{-i\xi}$ Fourier coefficients. This condition yields the equations

$$\frac{\partial p_0}{\partial \tau} - \frac{\partial p_0}{\partial \mu} = 0, \quad \frac{\partial q_0}{\partial \tau} + \frac{\partial q_0}{\partial \mu} = 0.$$

Thus a disturbance to p_0 propagates inwards at the linear-theory group velocity as τ increases (i.e. p_0 is associated with the incident wave). The normalization relative to the linear Airy-function solution implies that

$$|p_0| = \tfrac{1}{2}\pi^{-\frac{1}{2}},$$

In contrast, the τ dependence of $|q_0|$ is related to the unspecified transient, oscillatory or even unstable behaviour of the waves near the reflexion line.

Finally, to determine the slow phase shift due to the nonlinearity we need to examine the G_3 equation. The $e^{i\xi}$ non-secularity condition yields the equation

$$i\left(\frac{2}{3}\right)^{\frac{2}{3}}\left\{\frac{\partial p_1}{\partial \tau}-\frac{\partial p_1}{\partial \mu}\right\} = 2i\frac{\partial p_0}{\partial \chi}+\tfrac{4}{3}\beta(|p_0|^2+|q_0|^2)\,p_0.$$

The solution for p_1 is bounded if the long-term τ average of the terms on the right-hand side is zero. Although we do not know the detailed τ behaviour of $|q_0|^2$, we make the plausible assumption that its long-term average is the same as that of $|p_0|^2$ (i.e. that the long-term averages of the incident and reflected wave-action fluxes exactly balance). It then follows that

$$p_0 = \tfrac{1}{2}\pi^{-\frac{1}{2}}\exp\left(i\tfrac{1}{3}\beta\pi^{-1}\chi+i\,\text{constant}\right).$$

Translating back these results in terms of ρ, the time-independent incident wave has the form

$$\tfrac{1}{2}\pi^{-\frac{1}{2}}\exp\left(i\tfrac{2}{3}|\rho|^{\frac{3}{2}}+(i\beta/2\pi)\ln|\rho|+i\,\text{constant}\right) \quad \text{as} \quad \rho\to-\infty.$$

A condition which eliminates the possibly unsteady reflected wave is

$$|\rho|^{\frac{1}{4}}a+i|\rho|^{-\frac{1}{4}}a_\rho \sim \pi^{-\frac{1}{2}}\exp\left(i\tfrac{2}{3}|\rho|^{\frac{3}{2}}+(i\beta/2\pi)\ln|p|+i\,\text{constant}\right) \quad \text{as} \quad \rho\to-\infty.$$

This is the radiation condition used in §4. If there was forcing of the waves near the reflexion line then we should have to make do with the less precise radiation condition

$$\left||\rho|^{\frac{1}{4}}a+i|\rho|^{-\frac{1}{4}}a_\rho\right| \sim \pi^{-\frac{1}{2}} \quad \text{as} \quad \rho\to-\infty.$$

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